Heat kernel estimates for Schrödinger operators

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Outline

1. **Introduction**
2. **Main Results**
   - Unbounded potential V
   - Fractional Schrödinger with unbounded potential
   - V tends to 0 at infinity
3. **Proofs**

Gaussian estimate

A fundamental result by Aronson ('68) states that the fundamental solution (that is, the heat kernel) $p(t,x,y)$ of a second order uniformly parabolic equation in divergence form

$$\mathcal{L} = \sum_{i,j=1}^{d} \partial_j \left(\partial_i a_{ij}(x) \right)$$

on $\mathbb{R}^d$ enjoys the following Gaussian estimates

$$p(t,x,y) \asymp t^{-d/2} \exp\left(-\frac{|x-y|^2}{t} \right), \quad t > 0, x, y \in \mathbb{R}^d.$$

Schrödinger operator

- Let's consider the Schrödinger operator $\mathcal{L}_V = \Delta - V$ with non-negative potential $V \geqslant 0$;
- Let $\{T_t^V\}_{t\geqslant 0}$ be the semigroup associated with $\mathcal{L}$ with associated heat kernel $p(t,x,y)$, by Feynman-Kac formula we have

$$T_t^V f(x) = \mathbb{E}_x \left[f(B_t) \exp\left(- \int_0^t V(B_s)ds \right) \right],$$

where $\{B_t\}_{t\geqslant 0}$ denotes the standard d-dimensional Brownian motion.

Schrödinger operator

- If $0 < C_1 \leqslant V \leqslant C_2 < +\infty$ for some positive constants c_1, c_2, it is easy to see that

$$p(t,x,y) \asymp t^{-d/2} e^{-t} \exp\left(-\frac{|x-y|^2}{t}\right), \quad t > 0, x, y \in \mathbb{R}^d.$$

- V is in Kato class: Chung and Zhao ('94), Zhang ('97), Song ('06)., Li('08)

  Kato class:

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_0^t q(s,x,y)|V(y)|dyds = 0.$$

Schrödinger operator

Since then many efforts are devoted to characterize functions V for which $p(t,x,y)$ is comparable with the Gaussian estimates in space and time, i.e.

$$p(t,x,y) \asymp t^{-d/2} \exp\left(-\frac{|x-y|^2}{t}\right), \quad t>0, x,y \in \mathbb{R}^d.$$

- $V \in L^p(\mathbb{R}^d)$ with $p > d/2$: Simon('82), Semenov ('97), Liskevich and Semenov ('98), Bogdan, Dziubański, and Szczypkowski('19).
- $V \asymp (1+|x|)^{-m}$ with some $m > 2$: Zhang('00).
- Main methods: parabolic Harnack inequality, comparison principle, maximum principle.

Question: *Does there exist a global estimate on the heat kernel of $-\Delta + V$ different from Gaussian estimate, which reveals an explicit dependence on the potential V?*

- V is unbounded, i.e.

$$C_1 g(|x|) \leqslant V(x) \leqslant C_2 g(|x|)$$

for some $g : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{r\to+\infty} g(r) = +\infty$.
On diagonal estimates: Davies('89), Sikora('97),

$$p(t,x,x) \asymp \begin{cases} t^{-d/2} \exp\left(-tV(x)\right), & t \leqslant \frac{1+|x|}{\sqrt{g(|x|)}}, \\ e^{-t} \exp\left(-(1+|x|)\sqrt{V(x)}\right), & t > \frac{1+|x|}{\sqrt{g(|x|)}}. \end{cases}$$

The estimates on full regime is unknown.

Schrödinger operator

- Intrinsic ultracontractivity: Davies and Simon ('84):

$$C_1 e^{-\lambda_1 t}\varphi_1(x)\varphi_1(y) \leqslant p(t,x,y) \leqslant C_2 e^{-\lambda_1 t}\varphi_1(x)\varphi_1(y),$$

  where $\varphi_1(x)$ is a ground state (eigenfunction corresponding to the smallest eigenvalue λ_1) of the operator $\mathcal{L}$.

- $V = \pm\frac{\lambda}{|x|^2}$ (critical case): Zhang('00), Milman and Semenov ('04), Ishige, Kabeya and Ouhabaz('17)

$$p(t,x,y) \asymp t^{-d/2}\exp\left(-\frac{|x-y|^2}{t}\right)U(t,x)U(t,y)$$

  where $U(t,x) \asymp t^{\pm\sigma}$ when t is large enough and the constant σ will depend on λ.

Schrödinger operator

- $V = \pm\frac{\lambda}{|x|^\alpha}$ (critical case for stable process): Bogdan, Grzywny and Jakubowski('19), Cho, Kim, Song and Vondracek ('20), Jakubowski and Wang('20).
- $V \asymp (1+|x|)^{-m}$ with $m < 2$: Zhang('00): $p(t,x,y)$ is sub-exponential decay as $t \to \infty$ but the exact exponent is unknown.
- $V \geqslant -a(1+|x|)^{-2}$ with $a < a_1$: Zhang('01): upper bound $p(t,x,y)$ is polynomial growth as $t \to \infty$ but the exact exponent is unknown.

  $V \geqslant -a(1+|x|)^{-m}$ with $a > a_1$: Zhang('01):

$$p(t,x,y) \succeq q(t,x,y)\exp\left(C_1 t - \frac{C_2|x|^2}{t}\right).$$

Tools: probabilistic methods

Lemma

Let U be a domain of $\mathbb{R}^d$. Then, for every $x \in U$ and $y \notin \bar{U}$,

$$p(t,x,y) = \mathbb{E}_x\left[\exp\left(-\int_0^{\tau_U} V(B_s)ds\right) 1_{\{\tau_U \leqslant t\}} p\left(t-\tau_U, B_{\tau_U}, y\right)\right].$$

- This gives us a new way to obtain two-sided estimates of $p(t,x,y)$ by studying the behavior of $\{B_t\}_{t\geqslant 0}$ to visit the positions where V takes different values.
- Only the Feynman-Kac formua is applied in the proof of Lemma 1, so it still holds when $\{B_t\}_{t\geqslant 0}$ is replaced by more general strong Markovian processes.

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Tools: probabilistic methods

The following characterization for the distribution of (τ_D, B_{τ_D}) is proved by Hsu ('86).

Lemma (Hsu('86))

Suppose that D is a bounded domain with C^3 boundary. Then for every $x \in D$,

$$\mathbb{P}_x\left(\tau_D \in dt, B_{\tau_D} \in dz\right) = \frac{1}{2}\frac{\partial q_D(t,x,\cdot)}{\partial n}(z)\sigma(dz)\,dt,$$

where $\sigma(dz)$ denotes the Lebesgue surface measure on ∂D, and $\frac{\partial q_D(t,x,\cdot)}{\partial n}(z)$ denotes the exterior normal derivative of $q_D(t,x,\cdot)$ at the point $z \in D$ for Dirichlet heat kernel $q_D(t,x,y)$ associated with $\{B_t\}_{t\geqslant 0}$.

Tools: probabilistic methods

The following Levy-system are used to estimate for the distribution of (τ_D, X_{τ_D}) for jump process $\{X_t\}_{t\geqslant 0}$

Lemma

Suppose that $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ *is a nonnegative measurable function so that* $f(s,x,x) = 0$ *for all* $x \in \mathbb{R}^d$ *and* $s \in \mathbb{R}_+$*. It holds that*

$$\mathbb{E}_x\left[\sum_{s\leqslant\tau_D} f\left(s, X_{s-}, X_s\right)\right] = \mathbb{E}_x\left[\int_0^{\tau_D}\left(\int_{\mathbb{R}^d} f\left(s, X_s, z\right)\frac{C_{d,\alpha}}{|X_s - z|^{d+\alpha}}\,dz\right)ds\right],$$

where $X_{s-} := \lim_{t\to s: t>s} X_t$ *denotes the left limit of* $X_\cdot$ *at* s.

Tools

Lemma (Zhang('04))

For any given $C_0 > 0$ *there exist positive constants* C_i*,* $i = 1, \cdots, 4$*, so that for any* $R > 0$*,* $x \in \mathbb{R}^d$*,* $t \in (0, C_0R^2)$ *and* $y \in \partial B(x, R)$*,*

$$\frac{C_1 R}{t^{d/2+1}} \exp\left(-\frac{C_2 R^2}{t}\right) \leqslant \frac{\partial q_{B(x,R)}(t,x,\cdot)}{\partial n}(y) \leqslant \frac{C_3 R}{t^{d/2+1}} \exp\left(-\frac{C_4 R^2}{t}\right).$$

Lemma (Chen, Kim and Song('09))

For any constant $C_0 > 0$*, there exist positive constants* C_1 *and* C_2 *such that for every* $x \in \mathbb{R}^d$*,* $R > 0$*,* $y \in B(x, R)$ *and* $0 < t \leqslant C_0 R^\alpha$*,*

$$\tilde{q}_{B(x,R)}(t,x,y) \asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right)\left(\frac{\delta_{B(x,R)}(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right), \qquad (1)$$

where $\tilde{q}_{B(x,R)}(t,x,y)$ *denotes the Dirichlet heat kernel associated with* α*-stable process* $\{X_t\}_{t\geqslant 0}$.

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Setting

- Let $\mathcal{G}$ be the class of nondecreasing and strictly positive functions $g : [0, \infty) \to [1, +\infty)$ such that $\lim_{r\to\infty} g(r) = \infty$ and there exists a constant $C_0 > 0$ so that for all $r \geqslant 0$, $g(2r) \leqslant C_0 g(r)$. In this subsection we always assume that *there exist $g \in \mathcal{G}$ and positive constants C_1, C_2 so that for all $x \in \mathbb{R}^d$ with $|x| \geqslant 1$,*

$$C_1 g(|x|) \leqslant V(x) \leqslant C_2 g(|x|).$$

- For any $s \geqslant 0$, set

$$t_0(s) := \frac{1+s}{\sqrt{g(s)}}, \quad s > 0.$$

- We say $t_0 : [0, \infty) \to \mathbb{R}_+$ is almost increasing (resp. almost decreasing), if there exists an increasing (resp. a decreasing) function $h : [0, +\infty) \to [1, +\infty)$ such that

$$C_* h(s) \leqslant t_0(s) \leqslant C^* h(s), \quad s \geqslant 0.$$

Main results

Theorem (C.-Wang., 23+)

It holds for any $C_0 > 0$ that

(i) *for all $x, y \in \mathbb{R}^d$ and $0 < t \leqslant C_0 t_0(|x| \wedge |y|)$,*

$$p(t,x,y) \asymp t^{-d/2} \exp\left(-\frac{|x-y|^2}{t}\right) \times \exp\left(-\left(t(V(x)\wedge V(y)) + |x-y|\sqrt{V(x)\vee V(y)}\right)\right).$$

(ii) *for all $x, y \in \mathbb{R}^d$ and $t \geqslant C_0 t_0(|x| \wedge |y|)$,*

(ii-1) *if $s \mapsto t_0(s)$ is almost decreasing, then*

$$p(t,x,y) \asymp e^{-t} \exp\left(-((1+|x|)\sqrt{V(x)} + (1+|y|)\sqrt{V(y)})\right).$$

Main results

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Main results

Theorem (C.-Wang., 23+)

It holds for any $C_0 > 0$ *that*

(ii) *for all* $x, y \in \mathbb{R}^d$ *and* $t \geqslant C_0 t_0(|x| \wedge |y|)$,

(ii-2) *if* $s \mapsto t_0(s)$ *is almost increasing, then*

$$p(t,x,y) \asymp e^{-t} \min\left\{ \exp\left(-((1+|x|)\sqrt{V(x)} + (1+|y|)\sqrt{V(y)})\right), \exp\left(-\frac{(1+|x|)^2 + (1+|y|)^2}{t}\right)\right\}.$$

Example: $V(x) = |x|^\alpha$

Example

Assume that $V(x) = |x|^\alpha$ *for* $\alpha > 0$. *Let*

$$t_0(x,y) := \max\left\{(1+|x|)^{1-\alpha/2}, (1+|y|)^{1-\alpha/2}\right\}.$$

(i) *for every* $x, y \in \mathbb{R}^d$ *with* $|x| \leqslant |y|$ *satisfying* $|x-y| \leqslant |y|/2$,

$$p(t,x,y) \asymp \begin{cases} t^{-d/2} e^{-\frac{|x-y|^2}{t}} e^{-t(1+|y|)^\alpha}, & t \leqslant t_0(x,y), \\ e^{-t} e^{-(1+|y|)^{1+\alpha/2}}, & t > t_0(x,y). \end{cases} \quad (2)$$

(ii) *for every* $x, y \in \mathbb{R}^d$ *with* $|x| \leqslant |y|$ *satisfying* $|x-y| > |y|/2$,

$$p(t,x,y) \asymp \begin{cases} t^{-d/2} e^{-\frac{|x-y|^2}{t}} e^{-(1+|y|)^{1+\alpha/2}}, & t \leqslant t_0(x,y), \\ e^{-t} e^{-(1+|y|)^{1+\alpha/2}}, & t > t_0(x,y). \end{cases} \quad (3)$$

Application: Green's function estimates

Proposition

Let $G(x,y) := \int_0^\infty p(t,x,y)\,dt$ *be Green's function associated with the Schrödinger semigoup* $\{T_t^V\}_{t\geqslant 0}$*. Then, for all* $x,y\in\mathbb{R}^d$*,*

$$G(x,y) \asymp |x-y|^{-(d-2)}\cdot \Gamma_1(x,y)\Gamma_2(x,y),$$

where

$$\Gamma_1(x,y) = \exp\left(-|x-y|\sqrt{\max\{g(|x|),g(|y|)\}}\right).$$

$$\Gamma_2(x,y) = \begin{cases} 1, & d\geqslant 3,\\ \max\left\{\log\left(\frac{1}{|x-y|\sqrt{\max\{g(|x|),g(|y|)\}}}\right),1\right\}, & d=2,\\ \frac{1}{|x-y|\sqrt{\max\{g(|x|),g(|y|)\}}}, & d=1.\end{cases}$$

Application: Green's function estimates

- When $d \geqslant 2$,

$$G(x,y) \asymp \begin{cases} Q(x,y), & |x-y| \leqslant \frac{1}{\sqrt{\max\{g(|x|),g(|y|)\}}}, \\ |x-y|^{-(d-2)}\Gamma_1(x,y), & |x-y| > \frac{1}{\sqrt{\max\{g(|x|),g(|y|)\}}}, \end{cases}$$

  where $Q(x,y)$ denotes Green's function of the Laplacian operator Δ on $\mathbb{R}^d$;

- When $d = 1$, it holds that

$$G(x,y) \asymp \begin{cases} \frac{1}{\sqrt{\max\{g(|x|),g(|y|)\}}}, & |x-y| \leqslant \frac{1}{\sqrt{\max\{g(|x|),g(|y|)\}}}, \\ |x-y|^{-(d-2)}\Gamma_1(x,y), & |x-y| > \frac{1}{\sqrt{\max\{g(|x|),g(|y|)\}}}. \end{cases}$$

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$$\mathcal{L}^V = \Delta^{\alpha/2} - V.$$

Define

$$t_0(s) := \inf\left\{t > 0 : \exp\left(-tg(s)\right) = \frac{t}{(1+s)^\alpha}\right\}, \quad s \geqslant 0.$$

Theorem (Baraniewicz.-C.-Kaleta.-Wang.-Schilling., 23+)

Then there exists a constant $C_0 > 0$ such that for every $x, y \in \mathbb{R}^d$ with $|x| \leqslant |y|$ the following statements hold.

(1) *For every $0 < t \leqslant C_0 t_0(|x|)$, it holds that*

$$p(t,x,y) \asymp \left(\frac{1}{tg(|y|)} \wedge 1\right) e^{-tg(|x|)} \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right)$$

Theorem (Baraniewicz.-C.-Kaleta.-Wang.-Schilling., 23+)

(2) *If $t_0(\cdot)$ is almost non-increasing then for every $t > C_0 t_0(|x|)$,*

$$p(t,x,y) \asymp \frac{1}{g(|x|)g(|y|)(1+|x|)^{d+\alpha}(1+|y|)^{d+\alpha}} \cdot \int_{\{|z| \leqslant s_0(t)\}} e^{-C_{11} t g(|z|)} dz,$$

where $s_0(t) = h^{-1}(t) \vee 2$ with

$$h^{-1}(t) := \inf\{s > 1 : h(s) \leqslant t\}$$

If $t_0(\cdot)$ is almost non-decreasing, then for every $t > C_0 t_0(|x|)$,

$$p(t,x,y) \asymp \frac{e^{-t}}{g(|x|)g(|y|)(1+|x|)^{d+\alpha}(1+|y|)^{d+\alpha}}. \tag{4}$$

Example: $V(x) = \log^{\beta}(1+|x|)$

Example

Assume that $V(x) = \log^{\beta}(1+|x|)$ *for* $\beta > 0$. *Let*

$$t_0(x) := \log^{1-\beta}(1+|x|),\ \ \Psi(x) := \frac{1}{(1+|x|)^{d+\alpha}\log^{\beta}(1+|x|)}.$$

(i) *If* $\beta \in (1, +\infty)$, *for every* $x, y \in \mathbb{R}^d$ *with* $|x| \leqslant |y|$,

$$p(t,x,y) \asymp \begin{cases} \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right)\left(\frac{1}{t\log^{\beta}(1+|y|)} \wedge 1\right) e^{-t\log^{\beta}(1+|x|)}, & t \leqslant t_0(x), \\ e^{\left(\frac{1}{t}\right)^{\beta-1}}\Psi(x)\Psi(y), & t \in (t_0(x), 1], \\ e^{-t}\Psi(x)\Psi(y), & t \in (1, +\infty). \end{cases} \tag{5}$$

Example: $V(x) = \log^{\beta}(1+|x|)$

Example

(ii) If $\beta \in (0,1]$, for every $x, y \in \mathbb{R}^d$ with $|x| \leqslant |y|$,

$$p(t,x,y) \asymp \begin{cases} \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right)\left(\frac{1}{t\log^{\beta}(1+|y|)} \wedge 1\right) e^{-t\log^{\beta}(1+|x|)}, & t \leqslant t_0(x), \\ e^{-t}\Psi(x)\Psi(y), & t > t_0(x). \end{cases} \tag{6}$$

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Theorem (C.-Wang., 23+)

Suppose that there exist positive constants $m \in (0,2)$ and C_1, C_2 such that

$$C_1(1+|x|)^{-m} \leqslant V(x) \leqslant C_2(1+|x|)^{-m}, \quad x \in \mathbb{R}^d.$$

Let $t_0(x) := (1+|x|)^{1+\frac{m}{2}}$. Then for any $x, y \in \mathbb{R}^d$ with $|x| \geqslant |y|$ and $t > 0$,

$$p(t,x,y) \asymp \begin{cases} t^{-d/2} \exp\left(-\frac{|x-y|^2}{t}\right) \exp\left(-\frac{t}{(1+|x|)^m}\right), & t \leqslant t_0(|x|), \\ t^{-d/2} \exp\left(-\frac{|x-y|^2}{t}\right) \exp\left(-t^{\frac{2-m}{2+m}}\right), & t > t_0(|x|). \end{cases}$$

The case of negative potential V

Theorem (C.-Wang., 23+)

Suppose that there exist positive constants $m \in (0,2)$ and K_1, K_2 such that

$$-K_1(1+|x|)^{-m} \leqslant V(x) \leqslant -K_2(1+|x|)^{-m}, \quad x \in \mathbb{R}^d. \tag{7}$$

Then for any $x, y \in \mathbb{R}^d$ with $|x| \geqslant |y|$ and $t > 0$,

$$C_1 t^{-d/2} \exp\left(-\frac{C_2|x-y|^2}{t}\right) \exp\left(\max\left(\frac{C_3 t}{(1+|x|)^m}, C_3 t - \frac{C_4(1+|x|^2)}{t}\right)\right)$$
$$\leqslant p(t,x,y) \leqslant$$
$$C_5 t^{-d/2} \exp\left(-\frac{C_6|x-y|^2}{t}\right) \exp\left(\max\left(\frac{C_7 t}{(1+|x|)^m}, C_7 t - \frac{C_8(1+|x|^2)}{t}\right)\right) \tag{8}$$

The case of negative potential V

Theorem (C.-Wang., 23+)

- *Suppose that* (7) *holds for some* $m \in [2, +\infty)$ *and* $K_1, K_2 > 0$. *Then there exists a constant* $K_* > 0$ *so that, if* $K_1 \geqslant K_*$, *then the two-sided estimates* (8) *are still true for every* $x, y \in \mathbb{R}^d$ *and* $t > 0$.
- *Suppose that* $d \geqslant 3$, *and that* (7) *holds for some* $m \in (2, +\infty)$ *and* $K_1, K_2 > 0$. *Then there exists* $K^* > 0$ *such that, if* $K_2 \leqslant K^*$, *then for any* $x, y \in \mathbb{R}^d$ *and* $t > 0$,

$$p(t, x, y) \asymp q(t, x, y),$$

Upper bounds

Suppose that $0 < t \leqslant C_0 t_0(|x|)$ and $|x-y| \leqslant C_0' t^{1/2}$. By the semigroup property,

$$p(t,x,y) = \int_{\mathbb{R}^d} p\left(t/2,x,z\right) p\left(t/2,z,y\right) dz \leqslant c_1 t^{-d/2} \int_{\mathbb{R}^d} p\left(t/2,x,z\right) dz$$
$$= c_1 t^{-d/2} T^V_{t/2} 1(x) \leqslant c_2 t^{-d/2} \exp(-c_3 t g(|x|)).$$

Lemma

There are constants $C_1, C_2 > 0$ such that for all $t > 0$ and $x \in \mathbb{R}^d$

$$T^V_t 1(x) \leqslant C_1 \min\left\{ \exp\left(-C_2 t g(|x|)\right) + \exp\left(-C_2 \frac{(1+|x|)^2}{t}\right), e^{-C_2 t} \right\}.$$

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$$= c_1 t^{-d/2} T^V_{t/2} 1(x) \leqslant c_2 t^{-d/2} \exp(-c_3 t g(|x|)).$$

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Upper bounds

Suppose that $0 < t \leqslant C_0 t_0(|x|)$ and $|x-y| > 2C_0' t^{1/2}$. The proof is split into two cases.

Case 1: $|x-y| \leqslant |y|/4$. By the semigroup property we have

$$\begin{aligned} p(t,x,y) &= \int_{\{z:|z-y|\leqslant|x-y|/2\}} p\left(t/2,x,z\right) p\left(t/2,z,y\right) dz \\ &\quad + \int_{\{z:|z-y|>|x-y|/2\}} p\left(t/2,x,z\right) p\left(t/2,z,y\right) dz \\ &=: I_1 + I_2. \end{aligned}$$

When $|z-y| \leqslant |x-y|/2$, it holds that

$$|z-x| \geqslant |x-y| - |z-y| \geqslant \frac{|x-y|}{2} \geqslant C_0' t^{1/2}.$$

Then,

$$p\left(t/2,x,z\right) \leqslant c_1 \exp\left(-c_2\left(|x-z|\sqrt{\max\{g(|x|),g(|z|)\}} + \frac{|x-z|^2}{t}\right)\right).$$

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$$\begin{aligned} p(t,x,y) &= \int_{\{z:|z-y|\leqslant|x-y|/2\}} p\left(t/2,x,z\right) p\left(t/2,z,y\right) dz \\ &\quad + \int_{\{z:|z-y|>|x-y|/2\}} p\left(t/2,x,z\right) p\left(t/2,z,y\right) dz \\ &=: I_1 + I_2. \end{aligned}$$

When $|z-y| \leqslant |x-y|/2$, it holds that

$$|z-x| \geqslant |x-y| - |z-y| \geqslant \frac{|x-y|}{2} \geqslant C_0' t^{1/2}.$$

Then,

$$p\left(t/2,x,z\right) \leqslant c_1 \exp\left(-c_2\left(|x-z|\sqrt{\max\{g(|x|),g(|z|)\}} + \frac{|x-z|^2}{t}\right)\right).$$

Upper bounds

Lemma

Given any constant $C_0 > 0$, there exist positive constants C_1 and C_2 such that for all $x, y \in \mathbb{R}^d$ and $t > 0$ with $|x-y| > 2C_0 t^{1/2}$,

$$p(t,x,y) \leqslant C_1 t^{-d/2} \exp\left(-C_2\left(\frac{|x-y|^2}{t} + |x-y|\sqrt{\max\{g(|x|), g(|y|)\}}\right)\right)$$

Suppose that $|x-y| > 2C_0 t^{1/2}$. Set $U = B(x, |x-y|/3)$. Then,

$$p(t,x,y) = \mathbb{E}_x\left[\exp\left(-\int_0^{\tau_U} V(B_s)ds\right) 1_{\{\tau_U \leqslant t\}} p\left(t-\tau_U, B_{\tau_U}, y\right)\right]$$

$$\leqslant \mathbb{E}_x\left[\exp\left(-\tau_U \inf_{u\in U} V(u)\right) p\left(t-\tau_U, B_{\tau_U}, y\right)\right]$$

$$= \frac{1}{2}\int_0^t \exp\left(-s \inf_{u\in U} V(u)\right) \cdot \left(\int_{\partial U} p(t-s,z,y)\frac{\partial q_U(s,x,\cdot)}{\partial n}(z)\sigma(dz)\right) ds.$$

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Lemma

Suppose that there exist positive constants $m \in (0,2)$ *and* C_1, C_2 *such that*

$$C_1(1+|x|)^{-m} \leqslant V(x) \leqslant C_2(1+|x|)^{-m}, \quad x \in \mathbb{R}^d.$$

The following estimates hold.

$$T_t^V 1(x) \leqslant C_3 \left(\exp\left(-\frac{C_4 t}{(1+|x|)^m} \right) + \exp\left(-C_4 t^{\frac{2-m}{2+m}} \right) \right), \ \forall\, t > 0,\ x \in \mathbb{R}^d$$

Lemma

Suppose that there exist positive constants $m \in (2, \infty)$ *and* C_1, C_2 *such that*

$$-C_1(1+|x|)^{-m} \leqslant V(x) \leqslant -C_2(1+|x|)^{-m}, \quad x \in \mathbb{R}^d.$$

The following estimates hold.

$$\mathbb{E}_x\left[(1+|B_t|)^{-m}\right] \leqslant C_3 \begin{cases} \frac{\log(2+|x|)1_{\{d=m\}}}{(1+|x|)^{\min(d,m)}}, & t < (1+|x|)^2, \\ \frac{\left(\log(1+t)1_{\{d=m\}}\right)}{t^{\min(d,m)/2}}, & t \geqslant (1+|x|)^2. \end{cases}$$

Thank you for your attention!